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On a question of Gross

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Abstract

Using the notion of weighted sharing of sets we prove two uniqueness theorems which improve the results proved by Fang and Qiu [H. Qiu, M. Fang, A unicity theorem for meromorphic functions, Bull. Malaysian Math. Sci. Soc. 25 (2002) 31–38], Lahiri and Banerjee [I. Lahiri, A. Banerjee, Uniqueness of meromorphic functions with deficient poles, Kyungpook Math. J. 44 (2004) 575–584] and Yi and Lin [H.X. Yi, W.C. Lin, Uniqueness theorems concerning a question of Gross, Proc. Japan Acad. Ser. A 80 (2004) 136–140] and thus provide an answer to the question of Gross [F. Gross, Factorization of meromorphic functions and some open problems, in: Proc. Conf. Univ. Kentucky, Lexington, KY, 1976, in: Lecture Notes in Math., vol. 599, Springer, Berlin, 1977, pp. 51–69], under a weaker hypothesis.

Keywords: Meromorphic function; Uniqueness; Shared set; Weighted sharing

1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

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Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z\colon\ f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a \in S}\{z\colon\ f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In [2] Gross posed the following question:

Can one find two finite sets S_j $(j = 1, 2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Fang and Xu [1] considered the case of meromorphic functions and proved the following result.

Theorem A. [1] *Let $S_1 = \{z\colon\ z^3 - z^2 - 1 = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $\Theta(\infty; f) > \frac{1}{2}$ and $\Theta(\infty; g) > \frac{1}{2}$. If $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ then $f \equiv g$.*

Dealing with the question of Gross, Qiu and Fang [10] proved the following theorem.

Theorem B. [10] *Let $n \geqslant 3$ be a positive integer, $S_1 = \{0\}$, $S_2 = \{z\colon\ z^n - z^{n-1} - 1 = 0\}$ and let f and g be two nonconstant meromorphic functions whose poles are of multiplicities at least* 2. *If $E_f(\{\infty\}) = E_g(\{\infty\})$ and $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$ then $f \equiv g$.*

They also gave example to show that the condition that the poles of $f(z)$ and $g(z)$ are of multiplicities at least 2 cannot be removed in Theorem B.

It should be noted that if two meromorphic functions f and g have no simple pole then clearly $\Theta(\infty, f) \geqslant \frac{1}{2}$ and $\Theta(\infty, g) \geqslant \frac{1}{2}$.

Lahiri and Banerjee [8] investigated the situation for $\Theta(\infty, f) \leqslant \frac{1}{2}$ and $\Theta(\infty, g) \leqslant \frac{1}{2}$ in Theorem A and proved the following result.

Theorem C. [8] *Let $S_1 = \{z\colon\ z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n $(\geqslant 4)$ is an integer. If for two nonconstant meromorphic functions f and g $E_f(S_i) = E_g(S_i)$ for $i = 1, 2, 3$ and $\Theta(\infty; f) + \Theta(\infty; g) > 0$ then $f \equiv g$.*

Recently Yi and Lin [13] independently proved Theorem C assuming $\Theta(\infty; f) > 0$ instead of $\Theta(\infty; f) + \Theta(\infty; g) > 0$. They [13] further proved Theorem C also holds for $n = 3$ if one assumes $\Theta(\infty; f) > \frac{1}{2}$.

Yi and Lin [13] remarked that the assumption $E_f(S_2) = E_g(S_2)$ in the above results can be relaxed to $\overline{E}_f(S_2) = \overline{E}_g(S_2)$.

Now considering all the above theorems it is natural to ask the following question:

Is it possible in any way to further relax the nature of sharing the set S_1 in Theorem C?

In the present paper we shall investigate this problem and obtain two results which will improve all the previous theorems mentioned earlier. Also we shall provide an answer to the question of Gross in a more compact and convenient way than the previous authors have given.

To state our main result we shall take the aid of weighted sharing of values and sets as introduced in [5,6].

Definition 1. [5,6] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 2. [5] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

It should be mentioned that Lahiri [7] also gave an answer to the question of Gross in the case of meromorphic functions sharing two sets and proved the following theorem.

Theorem D. [7] *Let S_1 and S_3 be defined as in Theorem* C *and n $(\geqslant 7)$ is an integer. If for two nonconstant meromorphic functions f and g $E_f(S_1, 2) = E_g(S_1, 2)$, $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.*

In this paper we concentrate our investigation on uniqueness of meromorphic functions sharing three sets and as such a smaller value of n is expected.

We now state the following two theorems which are the main results of the paper.

Theorem 1. *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n $(\geqslant 4)$ is an integer. If for two nonconstant meromorphic functions f and g $E_f(S_1, 4) = E_g(S_1, 4)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 0$ then $f \equiv g$.*

The following example shows that the condition $\Theta(\infty; f) + \Theta(\infty; g) > 0$ is sharp in Theorem 1.

Example 1. Let

$$g = -a\frac{e^{(n-1)z} - 1}{e^{nz} - 1}, \qquad f(z) = e^z g(z)$$

and S_i's be as in Theorem 1. Then $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ because $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ and $f \equiv e^z g$. Also $\Theta(\infty; f) + \Theta(\infty; g) = 0$ and $f \not\equiv g$.

Theorem 2. *Let $S_1 = \{z\colon z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and n $(\geqslant 3)$ is an integer. If for two nonconstant meromorphic functions f and g $E_f(S_1, 6) = E_g(S_1, 6)$, $E_f(S_2, 0) = E_g(S_2, 0)$ and $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.*

Though for the standard definitions and notations of the value distribution theory we refer to [3], we now explain some notations which are used in the paper.

Definition 3. [4] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid =1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r,a;f \mid \leqslant m)$ $(N(r,a;f \mid \geqslant m))$ the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid \leqslant m)$ $(\overline{N}(r,a;f \mid \geqslant m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid <m)$, $N(r,a;f \mid >m)$, $\overline{N}(r,a;f \mid <m)$ and $\overline{N}(r,a;f \mid >m)$ are defined analogously.

Definition 4. We denote by $\overline{N}(r,a;f \mid = k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k, where $k \geqslant 2$ is an integer.

Definition 5. Let f and g be two nonconstant meromorphic functions such that f and g share (a,k) where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q$, by $\overline{N}_E^{(k+1}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geqslant k+1$; each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$ and $\overline{N}_E^{(k+1}(r,a;g)$.

Definition 6. [6] We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid \geqslant 2)$.

Definition 7. [5,6] Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 8. [9] Let $a,b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 9. [9] Let $a,b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

and

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}.$$

Lemma 1. [6, Lemma 1] *Let F, G be two nonconstant meromorphic functions sharing* $(1,1)$ *and $H \not\equiv 0$. Then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leqslant N(r,H)+S(r,F)+S(r,G).$$

Lemma 2. [9, Lemma 4] *If F, G share* $(1,0)$, $(\infty,0)$ *then*

$$\begin{aligned}N(r,H)\leqslant{}&\overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+\overline{N}_*(r,1;F,G)\\&+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 3. *Let f and g be two nonconstant meromorphic functions sharing* $(1,k)$, *where $2\leqslant k<\infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\cdots+(k-1)\overline{N}(r,1;f\mid=k)+k\overline{N}_L(r,1;f)\\&\quad+(k+1)\overline{N}_L(r,1;g)+k\overline{N}_E^{(k+1}(r,1;f)\leqslant N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

Proof. Let z_0 be a 1-point of f of multiplicity p and a 1-point of g of multiplicity q. We denote by $N_1(r)$, $N_2(r)$ and $N_3(r)$ the counting functions of those 1-points of f and g when $k+1\leqslant q<p$, $k+1\leqslant q=p$ and $k+1\leqslant p<q$, respectively. Each point in these counting functions is counted $q-k$ times.

Since f, g share $(1,k)$, we note that

$$\begin{aligned}N(r,1;g)-\overline{N}(r,1;g)={}&\overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\cdots\\&+(k-1)\overline{N}(r,1;f\mid=k)+(k-1)\overline{N}_L(r,1;f)\\&+(k-1)\overline{N}_L(r,1;g)+(k-1)\overline{N}_E^{(k+1}(r,1;f)\\&+N_1(r)+N_2(r)+N_3(r).\end{aligned}\tag{2.1}$$

Also we note that

$$N_1(r)\geqslant\overline{N}_L(r,1;f),\tag{2.2}$$
$$N_2(r)\geqslant\overline{N}_E^{(k+1}(r,1;f),\tag{2.3}$$
$$N_3(r)\geqslant 2\overline{N}_L(r,1;g).\tag{2.4}$$

Using (2.2)–(2.4) in (2.1) we deduce that

$$\begin{aligned}N(r,1;g)-\overline{N}(r,1;g)\geqslant{}&\overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\cdots\\&+(k-1)\overline{N}(r,1;f\mid=k)+k\overline{N}_L(r,1;f)\\&+(k+1)\overline{N}_L(r,1;g)+k\overline{N}_E^{(k+1}(r,1;f).\end{aligned}$$

This proves the lemma. □

Lemma 4. [8, Lemma 3] *Let f, g be two nonconstant meromorphic functions sharing* $(0,\infty)$, (∞,∞) *and $\Theta(\infty;f)+\Theta(\infty;g)>0$. Then $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ implies $f\equiv g$, where n $(\geqslant 2)$ is an integer and a is a nonzero finite constant.*

Lemma 5. [7, Lemma 5] *If two nonconstant meromorphic functions f, g share $(\infty, 0)$ then for $n \geqslant 2$,*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 6. [8, Lemma 7] *Let f, g be two nonconstant meromorphic functions sharing $(1, 2)$ and (∞, ∞). Then one of the following holds*:

(i) $T(r) \leqslant N_2(r, 0; f) + N_2(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g)$;
(ii) $f \equiv g$;
(iii) $fg \equiv 1$.

Lemma 7. [11] *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 8. *Let $F = \frac{f^{n-1}(f+a)}{-b}$, $G = \frac{g^{n-1}(g+a)}{-b}$ and $n \geqslant 3$ an integer. If f, g share $(0, 0)$, (∞, ∞), F, G share $(1, k)$ and $\Phi \not\equiv 0$ then*

$$\overline{N}(r, 0; g) = \overline{N}(r, 0; f) \leqslant \frac{1}{n-2}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).$$

Proof. Let z_0 be a zero of f and g. Then by a simple calculation we note that z_0 is a zero of Φ of multiplicity at least $n-2$. So using Lemma 7 and noting that F, G share (∞, ∞) we can write

$$\begin{aligned}(n-2)\overline{N}(r, 0; f) &\leqslant N(r, 0; \Phi)\\ &\leqslant T(r, \Phi)\\ &\leqslant N(r, \infty; \Phi) + S(r, F) + S(r, G)\\ &\leqslant \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g)\\ &\leqslant \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).\end{aligned}$$

Hence

$$\overline{N}(r, 0; f) \leqslant \frac{1}{n-2}\overline{N}_*(r, 1, F; G) + S(r, f) + S(r, g).$$

Since f and g share $(0, 0)$ implies $\overline{N}(r, 0; g) = \overline{N}(r, 0; f)$ the lemma follows from above. This completes the proof of the lemma. □

Lemma 9. [12, Lemma 6] *If $H \equiv 0$, then F, G share $(1, \infty)$.*

Lemma 10. *Let F, G share $(1, k)$, (∞, ∞) where $2 \leqslant k < \infty$ and $H \not\equiv 0$. Then*

$$\begin{aligned}T(r, F) \leqslant{}& N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) - m(r, 1; G)\\ &- \overline{N}(r, 1; F \mid = 3) - \cdots - (k-2)\overline{N}(r, 1; F \mid = k) - (k-2)\overline{N}_L(r, 1; F)\\ &- (k-1)\overline{N}_L(r, 1; G) - (k-1)\overline{N}_E^{(k+1}(r, 1; F) + S(r, F) + S(r, G).\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} T(r,F)+T(r,G) \leqslant\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\ &+\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_0(r,0;F')-N_0(r,0;G')\\ &+S(r,F)+S(r,G). \end{aligned} \tag{2.5}$$

In view of Definition 7, using Lemmas 1–3 we see that

$$\begin{aligned} \overline{N}(r,1;F)+\overline{N}(r,1;G) \leqslant\ & N(r,1;F\mid=1)+\overline{N}(r,1;F\mid=2)+\overline{N}(r,1;F\mid=3)+\cdots\\ &+\overline{N}(r,1;F\mid=k)+\overline{N}_E^{(k+1}(r,1;F)\\ &+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)\\ \leqslant\ & \overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+\overline{N}_*(r,\infty;F,G)\\ &+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid=2)+\cdots\\ &+\overline{N}(r,1;F\mid=k)+\overline{N}_E^{(k+1}(r,1;F)+\overline{N}_L(r,1;F)\\ &+\overline{N}_L(r,1;G)+T(r,G)-m(r,1;G)+O(1)\\ &-\overline{N}(r,1;F\mid=2)-2\overline{N}(r,1;F\mid=3)-\cdots\\ &-(k-1)\overline{N}(r,1;F\mid=k)-k\overline{N}_E^{(k+1}(r,1;F)\\ &-k\overline{N}_L(r,1;F)-(k+1)\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F')\\ &+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\ \leqslant\ & \overline{N}(r,0;F\mid\geqslant 2)+\overline{N}(r,0;G\mid\geqslant 2)+T(r,G)\\ &-m(r,1;G)-\overline{N}(r,1;F\mid=3)-2\overline{N}(r,1;F\mid=4)-\cdots\\ &-(k-2)\overline{N}(r,1;F\mid=k)-(k-2)\overline{N}_L(r,1;F)\\ &-(k-1)\overline{N}_L(r,1;G)-(k-1)\overline{N}_E^{(k+1}(r,1;F)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) in view of Definition 6 the lemma follows.

This proves the lemma. □

3. Proofs of the theorems

Proof of Theorem 1. Let $F=\frac{f^{n-1}(f+a)}{-b}$ and $G=\frac{g^{n-1}(g+a)}{-b}$. Then F and G share $(1,4)$, $(\infty;\infty)$. We consider the following cases.

Case 1. Suppose that $\Phi\not\equiv 0$. Then $F\not\equiv G$.

Subcase 1.1. Let $H\not\equiv 0$. Then by Lemmas 7 and 10 we get

$$\begin{aligned} nT(r,f) \leqslant\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &-2\overline{N}_L(r,1;F)-2\overline{N}_L(r,1;G)+S(r,f)+S(r,g)\\ \leqslant\ & 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+N_2(r,0;f+a)+N_2(r,0;g+a)\\ &+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)-2\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g). \end{aligned} \tag{3.1}$$

Hence for $\varepsilon>0$ we get by Lemma 8 from (3.1)

$$\begin{aligned}
nT(r,f) &\leqslant 4\overline{N}(r,0;f)+T(r,f)+T(r,g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&\quad -2\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant \frac{4}{n-2}\overline{N}_*(r,1;F,G)+T(r,f)+T(r,g)\\
&\quad +\bigl(1-\Theta(\infty;f)+\varepsilon\bigr)T(r,f)+\bigl(1-\Theta(\infty;g)+\varepsilon\bigr)T(r,g)\\
&\quad -2\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant \bigl[4-\Theta(\infty;f)-\Theta(\infty;g)+2\varepsilon\bigr]T(r)+S(r).
\end{aligned}\tag{3.2}$$

In the same way we can obtain

$$nT(r,g)\leqslant \bigl[4-\Theta(\infty;f)-\Theta(\infty;g)+2\varepsilon\bigr]T(r)+S(r).\tag{3.3}$$

Combining (3.2) and (3.3) we see that

$$\bigl[n-4+\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\bigr]T(r)\leqslant S(r),$$

which leads to a contradiction for $0<\varepsilon<\frac{\Theta(\infty;f)+\Theta(\infty;g)}{2}$. Hence this subcase does not hold.

Subcase 1.2. Let $H\equiv 0$. Then by Lemma 9, F, G share $(1,\infty)$ and hence by Lemma 8 we get $\overline{N}(r,0;f)=\overline{N}(r,0;g)=S(r,f)+S(r,g)$. If possible let us assume that the case (i) of Lemma 6 holds. Then by Lemma 7 and for $0<\varepsilon<\Theta(\infty;f)+\Theta(\infty;g)$ we get

$$\begin{aligned}
\max\bigl\{T(r,F),T(r,G)\bigr\} &= nT(r)+O(1)\\
&\leqslant N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)\\
&\quad +\overline{N}(r,\infty;G)+S(r,F)+S(r,G)\\
&\leqslant 4\overline{N}(r,0;f)+N_2(r,0;f+a)+N_2(r,0;g+a)\\
&\quad +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\
&\leqslant T(r,f)+T(r,g)+\left(1-\Theta(\infty;f)+\frac{\varepsilon}{2}\right)T(r,f)\\
&\quad +\left(1-\Theta(\infty;g)+\frac{\varepsilon}{2}\right)T(r,g)+S(r,f)+S(r,g),
\end{aligned}$$

that is,

$$\bigl[n-4+\Theta(\infty;f)+\Theta(\infty;g)-\varepsilon\bigr]T(r)\leqslant S(r),$$

which is a contradiction. So the case (i) of Lemma 6 does not hold.

Next if possible suppose that $FG\equiv 1$. Then by Lemma 5 we obtain a contradiction. Again since $F\not\equiv G$ this subcase also does not hold.

Case 2. Next suppose $\Phi\equiv 0$. Then by integration we get

$$F-1=c(G-1).\tag{3.4}$$

If possible let us assume $c\neq 1$. Then from (3.4) we obtain that 0 is a Picard exceptional value of f and g. Thus by the second fundamental theorem we get

$$\begin{aligned}T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1-c;F)+S(r,F)\\ &\leqslant \overline{N}(r,-a;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;G)+S(r,f)\\ &\leqslant \overline{N}(r,-a;f)+\overline{N}(r,-a;g)+\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

So using Lemma 7 we obtain

$$nT(r,f)+O(1)\leqslant 2T(r,f)+T(r,g)+S(r,f). \tag{3.5}$$

In a similar manner we can deduce

$$nT(r,g)+O(1)\leqslant T(r,f)+2T(r,g)+S(r,g). \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$nT(r)\leqslant 3T(r)+S(r). \tag{3.7}$$

Since $n\geqslant 4$, (3.7) leads to a contradiction.

Hence $c=1$. So $F\equiv G$, i.e., $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$. This together with the assumption that f and g share $(0,0)$ implies that f and g share $(0,\infty)$. Now the theorem follows from Lemma 4. This proves the theorem. □

Proof of Theorem 2. Let F and G be defined as in Theorem 1. Then F and G share $(1,6)$, $(\infty;\infty)$. We consider the following cases.

Case 1. Suppose that $\Phi\not\equiv 0$. Then $F\not\equiv G$.

Subcase 1.1. Let $H\not\equiv 0$. Then by Lemmas 7 and 10 we get

$$\begin{aligned}nT(r,f) &\leqslant N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &\quad -4\overline{N}_L(r,1;F)-4\overline{N}_L(r,1;G)+S(r,f)+S(r,g)\\ &\leqslant 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+N_2(r,0;f+a)+N_2(r,0;g+a)\\ &\quad +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)-4\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned} \tag{3.8}$$

Hence for $\varepsilon>0$ we get by Lemma 8 from (3.8)

$$\begin{aligned}nT(r,f) &\leqslant 4\overline{N}(r,0;f)+T(r,f)+T(r,g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\ &\quad -4\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\leqslant \frac{4}{n-2}\overline{N}_*(r,1;F,G)+T(r,f)+T(r,g)\\ &\quad +\big(1-\Theta(\infty;f)+\varepsilon\big)T(r,f)+\big(1-\Theta(\infty;g)+\varepsilon\big)T(r,g)\\ &\quad -4\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\leqslant \big[4-\Theta(\infty;f)-\Theta(\infty;g)+2\varepsilon\big]T(r)+S(r).\end{aligned} \tag{3.9}$$

In the same way we can obtain

$$nT(r,g)\leqslant \big[4-\Theta(\infty;f)-\Theta(\infty;g)+2\varepsilon\big]T(r)+S(r). \tag{3.10}$$

Combining (3.9) and (3.10) we see that

$$\big[n-4+\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon\big]T(r)\leqslant S(r),$$

which leads to a contradiction for $0 < \varepsilon < \frac{n-4+\Theta(\infty; f)+\Theta(\infty; g)}{2}$. Hence this subcase does not hold.

Subcase 1.2. Let $H \equiv 0$. Then by Lemma 9, F, G share $(1, \infty)$ and hence by Lemma 8 we get $\overline{N}(r, 0; f) = \overline{N}(r, 0; g) = S(r, f) + S(r, g)$. Now proceeding in the same way as in Subcase 1.2 in Theorem 1 we can show that this subcase does not hold.

Case 2. Next suppose $\Phi \equiv 0$. Then by integration we get

$$F - 1 = c(G - 1). \tag{3.11}$$

If possible let us assume $c \neq 1$. Then from (3.11) we obtain that 0 is a Picard exceptional value of f and g. Thus by the second fundamental theorem we get

$$\begin{aligned} T(r, F) &\leqslant \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1-c; F) + S(r, F) \\ &\leqslant \overline{N}(r, -a; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; G) + S(r, f) \\ &\leqslant \overline{N}(r, -a; f) + \overline{N}(r, -a; g) + \overline{N}(r, \infty; f) + S(r, f). \end{aligned}$$

So using Lemma 7 we obtain for $\varepsilon > 0$,

$$\begin{aligned} nT(r, f) &\leqslant T(r, f) + T(r, g) + \bigl\{1 - \Theta(\infty; f) + \varepsilon\bigr\} T(r, f) + S(r, f) \\ &\leqslant \bigl[3 - \Theta(\infty; f) + \varepsilon\bigr] T(r) + S(r). \end{aligned} \tag{3.12}$$

In the same way we can obtain

$$nT(r, g) \leqslant \bigl[3 - \Theta(\infty; g) + \varepsilon\bigr] T(r) + S(r). \tag{3.13}$$

From (3.12) and (3.13) we get

$$\bigl[n - 3 + \min\bigl\{\Theta(\infty; f), \Theta(\infty; g)\bigr\} - \varepsilon\bigr] T(r) \leqslant S(r)$$

which leads to a contradiction for $0 < \varepsilon < \min\{\Theta(\infty; f), \Theta(\infty; g)\}$. Hence $c = 1$ and the theorem follows from Lemma 4. This completes the proof of the theorem. □

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